A SCHLICHTNESS THEOREM FOR ENVELOPES OF HOLOMORPHY

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Abstract. Let Ω be a domain in $\mathbb{C}^2$. We prove the following theorem. If the envelope of holomorphy of Ω is schlicht over Ω, then the envelope is in fact schlicht. We provide examples showing that the conclusion of the theorem does not hold in $\mathbb{C}^n$, $n > 2$. Additionally, we show that the theorem cannot be generalized to provide information about domains in $\mathbb{C}^2$ whose envelopes are multiply sheeted.

1. Introduction

A distinct difference between complex analysis in one variable and in several variables is the existence in higher dimension of domains which are not Stein. We are led to consider the envelope of holomorphy of such a domain: the smallest Stein Riemann domain over $\mathbb{C}^n$ to which all functions holomorphic on our domain extend holomorphically. (See, for example [Gun90, Chapter H].)

In the case of domains in $\mathbb{C}^n$ with relatively simple structure, the envelope of holomorphy can, at least in principle, be calculated explicitly. Examples of such domains are Reinhardt, Hartogs, tubular and contractible domains. These classes of domains enjoy the distinct advantage that their envelopes are schlicht; that is, their envelopes are domains in $\mathbb{C}^n$.

In general, however, the envelope of a domain need not be schlicht. It may be a finitely or infinitely sheeted Riemann domain spread over $\mathbb{C}^n$. It is not obvious, given a description of a domain, whether its envelope is schlicht or multiply sheeted. Nor is it entirely clear how the geometry of a given domain relates to the geometry of its envelope. In fact, we have the following open question.

Problem 1.1. Does there exist a bounded domain, Ω, in $\mathbb{C}^n$, $n \geq 2$, such that Ω has smooth boundary, and the envelope of Ω is infinitely sheeted?

There are classical examples which show that there exist bounded domains whose envelopes are infinitely sheeted. Unfortunately, these domains have boundaries which are highly nonsmooth. (Such an example can be constructed, e.g., by nesting infinitely many Hartogs figures one inside the next, and carefully connecting them.)

An obvious problem suggested by the above discussion is that of determining why and when the envelope is schlicht or multiply sheeted. Can we, just by looking at a domain, determine whether its envelope is schlicht? Is there a way of controlling or understanding the number of sheets in the envelope of a domain? While we are unable to fully answer these questions, we present a theorem which indicates that

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in certain situations there are criteria which allow us to better understand whether envelopes are schlicht or not.

Specifically, we prove the following.

Theorem 1.2. Let Ω be a domain in $\mathbb{C}^2$, and let $(\widetilde{\Omega}, \pi)$ be its envelope of holomorphy. If π is injective on $\pi^{-1}(\Omega)$, then π is injective on $\widetilde{\Omega}$. In other words, if $\widetilde{\Omega}$ is schlicht over Ω, then $\widetilde{\Omega}$ is schlicht.

The theorem indicates that for domains in $\mathbb{C}^2$ there may be a relationship between the number of sheets in the envelope lying above the domain, and the number of sheets in the envelope. As we shall see in Section 3, there appears to be a relationship only in the special case of domains whose envelope is schlicht over the domain. We shall also see in Section 3 that there is no such relationship in higher dimensions. Further examples in this vein can be found in [Jup03].

We note that Chirka and Stout proved a weaker result in [CS95]. Specifically, they proved Lemma 2.6. Our methods are different, and better suited to proving and understanding the results in which we are interested.

2. A Schlichtness Theorem

We prove Theorem 1.2. Our proof proceeds as follows. Using the natural embedding of Ω into $\widetilde{\Omega}$, we are considering Ω as a subset of $\widetilde{\Omega}$ rather than as a domain in $\mathbb{C}^2$. Let Ω_1 be the set of points in $\widetilde{\Omega}$ which can be reached from Ω by pushing discs. Similarly, let Ω_n be the set of points in $\widetilde{\Omega}$ which can be reached from Ω_{n-1} by pushing discs. We recall that if two one-dimensional analytic varieties in $\mathbb{C}^2$ intersect nontrivially, then so do slight perturbations of these varieties. We proceed inductively, showing that π is injective on Ω_n for each n. In other words, we show that for each n, Ω_n can be identified with a domain in $\mathbb{C}^2$. To do so we push discs from Ω_{n-1}, keeping track of their intersections. Using the above fact about analytic varieties, we show that if Ω_n is not schlicht, then it is not schlicht over Ω_k, $k \leq n-1$. In particular Ω_n is not schlicht over Ω, contradicting the hypothesis that $\widetilde{\Omega}$ is schlicht over Ω.

We make several definitions which we shall need in the proof of the theorem.

Definition 2.1 (Pushing Discs)**.** We say that a point, $p \in \widetilde{\Omega}$, can be reached from Ω by pushing discs if there is a neighbourhood U of p in $\widetilde{\Omega}$ such that $\pi|_U$ is a biholomorphism, and such that the following holds: there is a biholomorphism, F, of $\Delta^2(0, 1)$ into U such that

(1) $p \in F(\Delta^2(0, 1))$, and
(2) $F(H) \subset\subset U \cap \Omega$,

where H is the Hartogs figure,

$$H = \left(\Delta(0, 1) \times \left\{\frac{1}{2} < |w| < 1\right\}\right) \cup \left(\Delta\left(0, \frac{1}{2}\right) \times \Delta(0, 1)\right).$$

Remark 2.2. Several observations and remarks about disc pushing should be made.

(1) We insist that our neighbourhoods, U, are biholomorphic to balls in $\mathbb{C}^2$, in particular $\pi(U)$ is a ball.
(2) The extension of a holomorphic function from $F(H)$ to $F(\Delta^2(0,\, 1))$ is single valued.
(3) Given a point, $p \in \widetilde{\Omega}$, which can be reached from Ω by pushing discs as above, we can assume (by rotation and scaling in the first coordinate, if necessary) that p is in $F(C)$, where

$$C = [0,\, 1] \times \Delta(0,\, 1).$$

In other words, we have a continuous family of holomorphic discs in $\widetilde{\Omega}$ such that the "bottom" disc and the boundaries of the discs lie in Ω, while the "top" disc is not contained in Ω..

Definition 2.3 (Ω_n)**.** We inductively define the sets Ω_n. Let

$$\Omega_0 = \Omega.$$

Let

$$\Omega_{n+1} = \{ \text{ all points in } \widetilde{\Omega} \text{ which can be reached from } \Omega_n \text{ by pushing discs } \}.$$

Remark 2.4. We make several observations about Ω_n.

(1) It is clear from the construction of Ω_n that these sets are open subsets of $\widetilde{\Omega}$.
(2) We have defined Ω_n as a subset of $\widetilde{\Omega}$, and when building Ω_n we push discs within $\widetilde{\Omega}$. However, if Ω_{n-1} is in fact a domain in $\mathbb{C}^2$, we can also view disc pushing as follows. We have

$$\pi(F(\Delta^2(0,\, 1))) \subset \mathbb{C}^2, \text{ where } \pi(F(H)) \subset \pi(\Omega_{n-1}) = \Omega_{n-1}.$$

In other words we push discs from Ω_{n-1}, considering Ω_{n-1} as a subset of $\mathbb{C}^2$. We then lift these discs to $\Omega_n \subset \widetilde{\Omega}$. By the identity principle for liftings ([JP00, Proposition 1.1.5]) the liftings are unique.

We next observe that we can use Ω_n to recover $\widetilde{\Omega}$.

Proposition 2.5. Let $\Omega' = \cup_{n=0}^{\infty} \Omega_n$. Then $\Omega' = \widetilde{\Omega}$.

Proof. Assume not, and let $p \in \widetilde{\Omega}$ be a point in $\partial\Omega'$. We claim that there is a neighbourhood U of p in $\widetilde{\Omega}$ such that $U \cap \Omega'$ is pseudoconvex.

In fact, let U be a neighbourhood of p such that $\pi|_U$ is a biholomorphism, and further choose U so that $\pi(U)$ is a ball. If $U \cap \Omega'$ is not pseudoconvex, then we can find a biholomorphism, F, of $\Delta^2(0,\, 1)$ into U such that

(1) $F(H) \subset\subset U \cap \Omega'$, and
(2) $F(\Delta^2(0,\, 1))$ contains points which are not in Ω'.

This follows from the fact that $\pi|_U$ is a biholomorphism, so that we can consider $U \cap \Omega'$ as a domain in $\mathbb{C}^2$, and from the notion of II-pseudoconvexity as described in [Pfl75, Chapter II.2].

Now $F(H) \subset\subset \Omega'$, so in fact $F(H) \subset\subset \Omega_n$, for some n. This implies that Ω_{n+1} contains points which are not in Ω'. This is a contradiction.

We conclude that every point in $\partial\Omega'$ has a neighbourhood, U, such that $U \cap \Omega'$ is pseudoconvex. By the equivalence of local and global pseudoconvexity of

unbranched Riemann domains over $\mathbb{C}^n$ (see e.g. [JP00, Corollary 2.2.16]) we see that Ω' is pseudoconvex. However, as Ω' is a subset of $\widetilde{\Omega}$, every holomorphic function on Ω extends in a single valued fashion to Ω'. We conclude that $\Omega' = \widetilde{\Omega}$. □ □

To prove Theorem 1.2 we require two lemmas.

Lemma 2.6. Assume that Ω_n is schlicht. Assume on the other hand that Ω_{n+1} is not schlicht. Then Ω_{n+1} is not schlicht over Ω_n.

Precisely, assume that there are points, $p_1 \neq p_2$ in Ω_{n+1} such that $\pi(p_1) = \pi(p_2)$. Then in fact there are points $q_1 \neq q_2$ in Ω_{n+1} such that $q = \pi(q_1) = \pi(q_2) \in \Omega_n$.

This lemma says, in particular, that if pushing discs from Ω once does not create any sheets over Ω, then in fact it does not create any sheets over Ω_1. As noted in the Introduction, Chirka and Stout have also proved this lemma [CS95].

Lemma 2.7. Assume that Ω_n is schlicht. If Ω_{n+1} is not schlicht over Ω_k, $k \leq n$, then Ω_{n+1} is not schlicht over Ω_{k-1}.

Once we have these lemmas we prove the theorem as follows.

Proof of Theorem 1.2. We inductively show that Ω_n is schlicht. By assumption we have that Ω_0 is schlicht.

If Ω_1 were not schlicht, then Lemma 2.6 would imply that it was not schlicht over Ω_0. This in turn means that $\widetilde{\Omega}$ is not schlicht over Ω_0. This is a contradiction, and we conclude that Ω_1 is schlicht.

Applying Lemma 2.6 to Ω_2 reveals that if Ω_2 were not schlicht, then it would not be schlicht over Ω_1. Lemma 2.7 now implies that Ω_2 is not schlicht over Ω_0. This again gives a contradiction, and we conclude that Ω_2 is schlicht.

We proceed similarly for Ω_n, $n \geq 3$. □ □

We now prove Lemmas 2.6 and 2.7.

Proof of Lemma 2.6. We begin by choosing two points, $p_1 \neq p_2$ in Ω_{n+1} with $p = \pi(p_1) = \pi(p_2)$. By definition and our comments after Definition 2.1, for each point p_j there is a neighbourhood, U_j, of p_j in $\widetilde{\Omega}$, and a biholomorphism F_j of $\Delta^2(0, 1)$ into $\widetilde{\Omega}$ satisfying

(1) $p_j \in F_j(C)$, and
(2) $F_j(H) \subset\subset U_j \cap \Omega_n$,

recalling that C is defined as

$$C = [0, 1] \times \Delta(0, 1).$$

We now define a subset c of C as

$$c = (\{0\} \times \Delta(0, 1)) \cup ([0, 1] \times \partial\Delta(0, 1)).$$

We have, by assumption, that $\pi(F_1(C)) \cap \pi(F_2(C))$ is nonempty; in particular it contains p. We let M be the connected component of $\pi(F_1(C)) \cap \pi(F_2(C))$ which contains p. We note that the $\pi(F_j(C))$ can be assumed to be in general position, so that their intersection is a two dimensional manifold. (The intersection is a closed set, so strictly speaking the intersection is a manifold with "ends".) By the stability of intersection of one-dimensional analytic manifolds in $\mathbb{C}^2$, we conclude that M contains a point, q, which lies in $\pi(F_1(c))$ or in $\pi(F_2(c))$. We then have

that $q = \pi(F_1(z_1))$ for some z_1 in c, or $q = \pi(F_2(z_2))$ for some z_2 in c. In the first case $q_1 = F_1(z_1)$ is in Ω_n, in the second case $q_2 = F_2(z_2)$ is in Ω_n.

We eliminate the possibility that both $q_1 = F_1(z_1)$ and $q_2 = F_2(z_2)$ are points in Ω_n. Otherwise, since Ω_n is schlicht and these two points have the same projection, we must have that $q_1 = q_2$. Let γ be a path in M connecting q to p. By the identity principle for liftings, we conclude that $p_1 = p_2$. This is a contradiction.

We examine the case where $z_1 \in c$, hence $F_1(z_1)$ is in Ω_n. Since $q_2 = F_2(z_2)$ is not in Ω_n, $q_1 \neq q_2$. But $\pi(q_1) = \pi(q_2)$. We now conclude that Ω_{n+1} is not schlicht over Ω_n.

The case where q is in $\pi(F_2(c))$ but not in $\pi(F_1(c))$ proceeds analogously. □ □

The proof of Lemma 2.7 is similar.

Proof of Lemma 2.7. We begin by choosing two points, $p_1 \neq p_2$, with p_1 in Ω_{n+1} and p_2 in Ω_k, and with $p = \pi(p_1) = \pi(p_2)$. By definition, for each point p_j there is a neighbourhood, U_j, of p_j in $\widetilde{\Omega}$, and a biholomorphism F_j of $\Delta^2(0, 1)$ into $\widetilde{\Omega}$ satisfying

(1) $p_1 \in F_1(C)$, $F_1(H) \subset\subset U_1 \cap \Omega_n$, and
(2) $p_2 \in F_2(C)$, $F_2(H) \subset\subset U_2 \cap \Omega_{k-1}$.

As in the proof of Lemma 2.6, we have that $\pi(F_1(C)) \cap \pi(F_2(C))$ is nonempty; in particular it contains p. We let M be the connected component of $\pi(F_1(C)) \cap \pi(F_2(C))$ which contains p. By the stability of intersection of one-dimensional analytic manifolds in $\mathbb{C}^2$, we conclude that M contains a point, q, which lies in $\pi(F_1(c))$ or in $\pi(F_2(c))$.

It is not possible that $F_1(z_1)$ is in Ω_n. Indeed, since $q_2 = F_2(z_2)$ is in $\Omega_k \subset \Omega_n$, this would imply that we have $F_1(z_1) = F_2(z_2)$ by the fact that Ω_n is schlicht. As above, the path $\gamma \subset M$ connecting q and p yields the contradiction that $p_1 = p_2$.

We thus must have that $q_1 = F_1(z_1)$ is in Ω_{n+1} but not in Ω_n, and that $q_2 = F_2(z_2) \in F_2(c)$ and hence it is in Ω_{k-1}. This means that Ω_{n+1} is not schlicht over Ω_{k-1}. With this, the proof of the lemma is complete. □ □

2.1. **Remarks.** Theorem 1.2 also holds for two dimensional Riemann domains over $\mathbb{C}^2$. The proof proceeds exactly as above.

3. Counterexamples

3.1. **Counterexamples in $\mathbb{C}^2$.** We construct a domain, Ω, in $\mathbb{C}^2$ such that the envelope of Ω has two sheets over Ω, but three sheets over $\mathbb{C}^2$. This shows that Theorem 1.2 cannot be generalized to give information about domains in $\mathbb{C}^2$ whose envelopes are multiply sheeted.

The example is obtained as follows. We construct three families of analytic discs whose intersection is a small set. For each family we find a pseudoconvex domain close to the family, making sure that certain boundary points of the new domain are in fact strictly pseudoconvex. We call this new domain a fattening of the family. We then build a frame for the family: a small neighbourhood of the union of the bottom disc of the family and the boundaries of the discs in the family. Pushing discs in this frame gives us the fattened family with which we started. We join the frames with well chosen paths and find pseudoconvex neighbourhoods of the paths.

The domain, Ω, is the union of the frames and the paths. The correct choice of paths ensures that our domain has the desired properties.

We begin by constructing three families of discs,

$$\begin{aligned} \Sigma_s &: \Delta(0,\, 2) \to \mathbb{C}^2, && s \in (-1,\, \epsilon_\Sigma), \\ \Delta_t &: \Delta(0,\, 1/2) \to \mathbb{C}^2, && t \in (-\epsilon_\Delta,\, 1), \\ \Gamma_l &: \Delta(0,\, r_\Gamma) \to \mathbb{C}^2, && l \in (-\epsilon_\Gamma,\, \epsilon_\Gamma) \end{aligned}$$

where ϵ_Σ, ϵ_Δ, ϵ_Γ and r_Γ are small positive real numbers to be carefully chosen, and the maps are defined as

$$\begin{aligned} \Sigma_s(w) &= (s,\, w), \\ \Delta_t(z) &= (z,\, t), \\ \Gamma_l(\xi) &= (\xi + i\eta\xi^2,\, \xi - il), \end{aligned}$$

with η a small positive real number to be carefully chosen.

We shall refer to the families as Σ, Δ and Γ. By the bottom disc of the family Σ, Δ or Γ we mean, respectively, the discs Σ_{-1}, Δ_1 or Γ_{ϵ_Γ}. For ease of notation we denote these discs by Σ_B, Δ_B and Γ_B respectively. By the boundary of the family Σ, Δ or Γ we mean, respectively, the family of circles

$$\begin{aligned} \Sigma_s|_{\partial\Delta(0,\,2)}, &\quad s \in (-1,\, \epsilon_\Sigma), \\ \Delta_t|_{\partial\Delta(0,\,1/2)}, &\quad t \in (-\epsilon_\Delta,\, 1), \\ \Gamma_l|_{\partial\Delta(0,\,r_\Gamma)} &\quad l \in (-\epsilon_\Gamma,\, \epsilon_\Gamma). \end{aligned}$$

We denote these boundaries by $\partial\Sigma$, $\partial\Delta$, and $\partial\Gamma$, respectively. Finally, $\Sigma' = \Sigma \cup \partial\Sigma$, $\Delta' = \Delta \cup \partial\Delta$, and $\Gamma' = \Gamma \cup \partial\Gamma$. Let π_j be the projection onto the jth coordinate of $\mathbb{C}^2$.

For each of the three following claims we must make suitable choices of ϵ_Σ, ϵ_Δ, ϵ_Γ, r_Γ and η.

We would first like to show that the intersection of the three families, $\Sigma \cap \Delta \cap \Gamma$, is a small set. We let

$$P = \Bigg\{ (z_1,\, z_2) \in \mathbb{C}^2\,;\, \mathrm{Im}\ z_1 = \mathrm{Im}\ z_2 = 0, \\ \frac{-1}{2} \leq \mathrm{Re}\ z_1 \leq \epsilon_\Sigma,\, -\epsilon_\Delta \leq \mathrm{Re}\ z_2 \leq 1 \Bigg\}.$$

We see that

$$\Sigma' \cap \Delta' = P.$$

Let G be the union of the graphs of the functions

$$f(y) = \pm\left(y^2 - \frac{y}{\eta}\right)^{1/2},\, y \leq 0.$$

In other words

$$G = \{(x,\, y) \in \mathbb{R}^2\,;\, (f(y),\, y),\, y \leq 0\}.$$

A point in Γ_l looks like $(\xi + i\eta\xi^2,\, \xi - il)$. Let $\xi = x + iy$.

Choosing $r_\Gamma < 1/\eta$ we see that for a point, $\Gamma_l(\xi) = (\xi + i\eta\xi^2,\, \xi - il)$, in Γ_l to be in P we must have that

(1) $y \leq 0$,
(2) $y = l$, and
(3) $x = \pm\left(y^2 - \frac{y}{\eta}\right)^{1/2}$.

Thus $\Gamma_l(\xi) = \Gamma_y(\xi)$. If $l > 0$ and $\xi \in \Delta(0, r_\Gamma)$ then $\Gamma_l(\xi)$ is not in P. If $l = 0$ and $\Gamma_l(\xi)$ is in P, then $\xi = 0$. Finally, if $l < 0$ and $\Gamma_l(\xi)$ is in P, then $\xi = \pm(l^2 - l/\eta)^{1/2} + il$.

We have shown:

Claim 3.1. The intersection of the three families, $\Sigma \cap \Delta \cap \Gamma$, is a small set. Specifically,

$$\Sigma \cap \Delta \cap \Gamma = P \cap \{\Gamma_y(\xi)\,;\, \xi = x + iy \in G \cap \Delta(0, r_\Gamma),\, -\epsilon_\Gamma \leq y \leq 0\}.$$

The following claim will allow us to choose the frames of our families to be disjoint.

Claim 3.2. The boundaries of the families are pairwise disjoint. Similarly, the bottom discs of the families are pairwise disjoint. The boundary of each family is disjoint from the bottom disc of the other families.

Finally, we note that the boundaries and bottoms of our families of discs are disjoint from the intersection of the three families.

Claim 3.3. The sets Σ_B, Δ_B, Γ_B, $\partial\Sigma$, $\partial\Delta$ and $\partial\Gamma$ are all disjoint from $\Sigma \cap \Delta \cap \Gamma$.

Given a family of discs such as Δ, and any $\delta > 0$, we can find a pseudoconvex domain, $\Delta^{\square}$, contained in a δ neighbourhood of Δ. We can then construct a subdomain, $\Delta^{\sqcup}$, of $\Delta^{\square}$ such that

(1) $\Delta^{\sqcup}$ is contained within a δ neighbourhood of $\Delta_B \cup \partial\Delta$,
(2) $\widetilde{\Delta}^{\sqcup} = \Delta^{\square}$, and
(3) the boundary of $\Delta^{\sqcup}$ contains smooth points which consist of strictly pseudoconvex points.

We call $\Delta^{\square}$ a fattening of Δ, and we call $\Delta^{\sqcup}$ the frame of Δ.

An analogous construction can be carried out for Γ and Σ. In fact, by Claim 3.2 we can choose these frames to be pairwise disjoint. By Claim 3.3 we can also choose them so that they do not intersect $\Sigma^{\square} \cap \Delta^{\square} \cap \Gamma^{\square}$.

Let γ_1 and γ_2 be two disjoint smooth paths, $\gamma_1 : [0, 1] \to \mathbb{C}^2$ and $\gamma_2 : [0, 1] \to \mathbb{C}^2$, satisfying:

(1) The path γ_1 connects a strictly pseudoconvex boundary point of $\Delta^{\sqcup}$ and a strictly pseudoconvex boundary point of $\Sigma^{\sqcup}$. Similarly, γ_2 connects a strictly pseudoconvex boundary point of $\Sigma^{\sqcup}$ and a strictly pseudoconvex boundary point of $\Gamma^{\sqcup}$.
(2) The intersection of $\Sigma^{\square} \cup \Delta^{\square} \cup \Gamma^{\square}$ and $\gamma_1 \cup \gamma_2$ consists of the four endpoints of the paths.
(3) The curve γ_1 is transversal to the boundaries of $\Delta^{\sqcup}$ and $\Sigma^{\sqcup}$ at its endpoints. The analogous statement holds for γ_2.
(4) Let p be a point in the complement of $\pi_1(\Sigma^{\square} \cup \Delta^{\square} \cup \Gamma^{\square})$, and define the set A as
$$A = \mathbb{C}^2 \backslash (\{z_1 = p\} \times \mathbb{C}).$$
The fundamental group of A is nontrivial. We choose γ_1 in such a way that

(a) $\Delta^{\sqcup} \cup \gamma_1 \cup \Sigma^{\sqcup}$ contains no nontrivial element of the fundamental group, and
(b) $\Delta^{\square} \cup \gamma_1 \cup \Sigma^{\square}$ does contain a nontrivial element of the fundamental group.

We choose γ_2 so that analogous statements hold for γ_2, Σ and Γ.

Item 4 implies that $f(z_1, z_2) = (z_1 - p)^{1/3}$ is holomorphic on a small neighbourhood of

$$\Sigma^{\sqcup} \cup \Delta^{\sqcup} \cup \Gamma^{\sqcup} \cup \gamma_1 \cup \gamma_2,$$

but triple valued on a small neighbourhood of

$$\Sigma^{\square} \cup \Delta^{\square} \cup \Gamma^{\square} \cup \gamma_1 \cup \gamma_2.$$

Indeed, as we follow γ_1 from $\Delta^{\square}$ to $\Sigma^{\square}$ we change branches of the cube root function. Similarly, as we follow γ_2 from $\Sigma^{\square}$ to $\Gamma^{\square}$ we change branches again.

By a theorem of Fornæss and Stout [FS77] we can find a neighbourhood, Γ_1, of γ_1 such that Γ_1 is pseudoconvex, $\Gamma_1 \cup \Delta^{\sqcup}$ is locally pseudoconvex near the intersection of γ_1 and $\Delta^{\sqcup}$, and $\Gamma_1 \cup \Sigma^{\sqcup}$ is locally pseudoconvex near the intersection of γ_1 and $\Sigma^{\sqcup}$. Similarly, we find a neighbourhood, Γ_2, of γ_2 with analogous properties. These neighbourhoods can be made as small as we like.

We define Ω as

$$\Omega = \Sigma^{\sqcup} \cup \Delta^{\sqcup} \cup \Gamma^{\sqcup} \cup \Gamma_1 \cup \Gamma_2.$$

As noted above, $f(z_1, z_2) = (z_1 - p)^{1/3}$ is holomorphic on Ω, but triple valued on $\Sigma^{\square} \cup \Delta^{\square} \cup \Gamma^{\square} \cup \Gamma_1 \cup \Gamma_2$. We conclude that the envelope of Ω is triple sheeted. Over Ω, however, the envelope is only double sheeted, specifically over those points in the pairwise intersections of our three families of discs. In fact, the envelope of Ω can be identified with the Riemann domain given as the disjoint union of $\Sigma^{\square}$, $\Delta^{\square}$ and $\Gamma^{\square}$, connected by the sets Γ_1 and Γ_2, equipped with the natural projection. Certainly every holomorphic function on Ω extends to this Riemann domain. By construction it is an unbranched Riemann domain which is locally pseudoconvex. By the equivalence of local and global pseudoconvexity of unbranched Riemann domains ([JP00, Corollary 2.2.16]), this Riemann domain is Stein, and thus is the envelope of Ω.

3.2. **Counterexamples in $\mathbb{C}^3$.** We construct a domain, Ω, in $\mathbb{C}^3$ such that the envelope of Ω has one sheet over Ω, but two sheets over $\mathbb{C}^3$. This shows that Theorem 1.2 cannot be generalized to give information about domains in $\mathbb{C}^n$, $n > 2$.

The key point in the example is that the intersection of one dimensional varieties in $\mathbb{C}^3$ is not generically preserved under perturbation.

We define Ω as follows. We first build two domains, V_1 and V_2, with strictly pseudoconvex boundary points. We join these domains with a well chosen path γ. We let π_j be the projection onto the jth coordinate of $\mathbb{C}^3$.

Begin by defining two domains, U_1 and U_2. Let U_1 be

$$U_1 = \left[\{|z| < 8\} \times \left\{\frac{1}{2} < |w| < 1\right\} \times \left\{|\zeta| < \frac{1}{2}\right\}\right] \cup \left[\left\{\frac{1}{2} < |z| < 8\right\} \times \left\{\frac{1}{2} < |w| < 1\right\} \times \{|\zeta| < 1\}\right].$$

We see that $\widetilde{U}_1$ is

$$\widetilde{U}_1 = \{|z| < 8\} \times \left\{\frac{1}{2} < |w| < 1\right\} \times \{|\zeta| < 1\}.$$

Let U_2 be

$$U_2 = \left[\left\{|z| < \frac{1}{4}\right\} \times \{|w| < 8\} \times \left\{\frac{3}{2} < |\zeta| < 2\right\}\right] \cup \left[\left\{|z| < \frac{1}{4}\right\} \times \left\{\frac{3}{2} < |w| < 8\right\} \times \left\{\frac{3}{4} < |\zeta| < 2\right\}\right].$$

We see that $\widetilde{U}_2$ is

$$\widetilde{U}_2 = \left\{|z| < \frac{1}{4}\right\} \times \{|w| < 8\} \times \left\{\frac{3}{4} < |\zeta| < 2\right\}.$$

Notice that

$$U_1 \cap U_2 = \emptyset, \quad \widetilde{U}_1 \cap U_2 = \emptyset, \quad \widetilde{U}_2 \cap U_1 = \emptyset,$$

but that $\widetilde{U}_1 \cap \widetilde{U}_2$ is non empty.

We define

$$V_1 = U_1 \cap B^3(0, 6)$$

and

$$V_2 = U_2 \cap B^3(0, 6).$$

Just as with U_1 and U_2 we have that

$$V_1 \cap V_2 = \emptyset, \quad \widetilde{V}_1 \cap V_2 = \emptyset, \quad \widetilde{V}_2 \cap V_1 = \emptyset,$$

but that $\widetilde{V}_1 \cap \widetilde{V}_2$ is non empty.

Unlike U_j, however, each V_j has strictly pseudoconvex boundary points: points in $\partial V_j \cap \partial B^3(0,\,6)$. Let p_1 and p_2 be such boundary points in V_1 and V_2, respectively. Let $\gamma : [0,\,1] \to \mathbb{C}^3$ be a smooth curve satisfying:

(1) γ runs from p_1 to p_2.
(2) γ is transversal to ∂V_j at p_j.
(3) γ intersects $\overline{B^3(0, 6)}$ only at p_1 and p_2.
(4) Let p be a point in the complement of $\pi_1(B^3(0,\,6))$, and define the set

$$A = \mathbb{C}^3 \backslash (\{z = p\} \times \mathbb{C}^2).$$

The fundamental group of A is nontrivial. We choose γ in such a way that
 (a) $V_1 \cup V_2 \cup \gamma$ contains no nontrivial element of the fundamental group, and
 (b) $\widetilde{V}_1 \cup \widetilde{V}_2 \cup \gamma$ does contain a nontrivial element of the fundamental group.

As in the previous example, we find a neighbourhood, Γ, of γ such that Γ is pseudoconvex, $\Gamma \cup V_1$ is locally pseudoconvex near p_1, and $\Gamma \cup V_2$ is locally pseudoconvex near p_2.

We define Ω as $\Omega = V_1 \cup V_2 \cup \Gamma$.

Our choice of path ensures that $f(z, w, \zeta) = (z - p)^{1/2}$ is a holomorphic function on Ω. However, f is not holomorphic on $\widetilde{V}_1 \cup \widetilde{V}_2 \cup \Gamma$: as we follow γ from $\widetilde{V}_1$ to $\widetilde{V}_2$ we change branches of the square root function. We conclude that f is double valued on $\widetilde{V}_1 \cup \widetilde{V}_2 \cup \Gamma$. Thus the envelope of Ω is double sheeted. The two sheets lie over $\widetilde{V}_1 \cap \widetilde{V}_2$. Since this intersection contains no points in Ω, we see that the

envelope is single sheeted over Ω. As in the previous example, we are viewing the envelope as a particular unbranched Riemann domain: the disjoint union of $\widetilde{V}_1$ and $\widetilde{V}_1$, connected with the set Γ, and equipped with the natural projection.

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